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The black hole dynamical horizon and generalized second law of thermodynamics

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Abstract: The generalized second law of thermodynamics for a system containing a black hole dynamical horizon is proposed in a covariant way. Its validity is also tested in case of adiabatically collapsing thick light shells.

Keywords: Space-Time Symmetries, Black Holes, Classical Theories of Gravity, Spacetime Singularities.

Contents

1. Introduction

The generalized second law of thermodynamics was initially put forth for a system including black holes by Bekenstein[1, 2, 3]. It states that the sum of one quarter of the area of the black hole's event horizon plus the entropy of ordinary matter outside never decreases with time in all processes. It is noteworthy that for the formation or absorption of black holes the generalized second law of thermodynamics can also be equivalently formulated as a covariant entropy bound. Namely, the entropy flux S through the event horizon between its two-dimensional space-like surfaces of area A_e and A'_e must satisfy

$$S \leq \frac{A'_e - A_e}{4}, \tag{1.1}$$

where $A'_e \geq A_e$ is assumed.

However, due to the global and teleological property of event horizon, the notion of dynamical horizon was developed and its properties were investigated, where, in particular, the first and second laws of black hole mechanics was generalized to the dynamical horizon[4, 5, 6, 7]. Thus it is tempting to conjecture that the dynamical horizon may also have the thermal character as the event horizon does, and the generalized second law of thermodynamics may also be applied to the dynamical horizon. This is what we shall address in the present paper. In next section, we shall propose a covariant entropy bound formulation of the generalized second law of thermodynamics associated with the black hole dynamical horizon. Then its validity is demonstrated in a model

of a growing black hole by spherically symmetrical collapse of thick light shells. Some discussions are presented in the end.

The signature of metric takes $(-,+,+,+)$. Notation and conventions follow Ref.[8].

2. The generalized second law of thermodynamics associated with the black hole dynamical horizon

We would first like to introduce the basic definition of the black hole dynamical horizon. For more subtle details, please refer to Ref. [5] and references therein.

Definition: A smooth, three-dimensional, space-like sub-manifold in a space-time (M, g_{ab}) is said to be a black hole dynamical horizon if it can be foliated by a family of closed two-dimensional surfaces such that, on each leaf, the expansion θ_l of one future-directed null normal l^a vanishes and the expansion θ_n of the other future-directed null normal n^a is strictly negative. If we choose the normalization of l^a and n^a such that $l^a n_a = -2$, then the expansion of the null geodesics normal can be given by $\theta_l = h^{ab}\nabla_a l_b (\theta_n = h^{ab}\nabla_a n_b)$ with the induced metric $h_{ab} = g_{ab} + \frac{1}{2}(l_a n_b + n_a l_b)$ on each leaf.

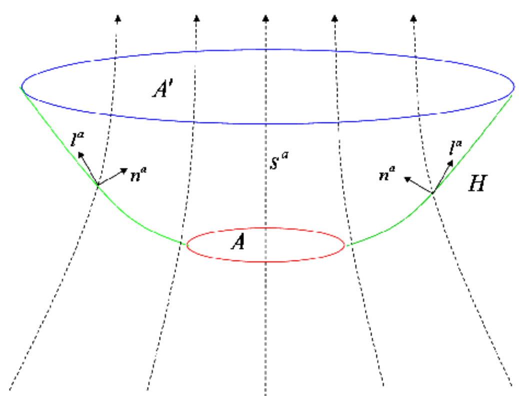

Figure 1: *A black hole dynamical horizon H between its apparent horizon of area A and A' with entropy current s^a flowing through it.*

Thus, roughly speaking, a black hole dynamical horizon is a space-like hyper-surface which is foliated by closed apparent horizons, where l^a and n^a represent future-directed outgoing and ingoing null normals, respectively. See Fig.1. Note that, in contrast to the notion of the event horizon, the dynamical horizon can be identified quasi-locally without knowledge of the full space-time history. In addition, intuitively, it is clear that no signal can propagate out of the dynamical horizon due to the fact that the dynamical horizon is space-like. All of these make the dynamical horizon become a competent candidate for the boundary of the black hole.

Now associated with the alternative boundary of the black hole, the generalized second law of thermodynamics can be naturally formulated in a covariant way as follows: *The entropy flux S through the black hole dynamical horizon between its apparent horizons of area A and A' must satisfy $S \leq \frac{A'-A}{4}$ if the dominant energy condition holds for matter, where $A' > A$ is assumed.*

In the subsequent section, its validity will be tested by adiabatically collapsing thick light shells.

3. The generalized second law of thermodynamics tested by adiabatically collapsing thick light shells

Start with a model of formation of a black hole by spherically symmetrical collapse of thick light shells in Eddington-Finkelstein coordinate[14]

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 + u(dt + dr)^2, \tag{3.1}$$

where

$$u = \begin{cases} 0, & \text{Minkowski region,} \\ \frac{2m}{r}F(\frac{r+t}{\Delta}), & \text{within light shells,} \\ \frac{2m}{r}, & \text{Schwarzschild region.} \end{cases} \tag{3.2}$$

Here both m and Δ are constant parameters. In addition, F is a monotonically increasing function with respect to its argument, with $F(0) = 0$, and $F(1) = 1$. Thus F essentially serves as a density profile function. After a straightforward but tedious calculation, one finds that this metric is a solution of Einstein equation with the non-vanishing energy momentum tensor within light shells being given by

$$T_{ab} = \frac{mF'}{4\pi\Delta r^2}k_a k_b, \tag{3.3}$$

where F' denotes the derivative of F with respect to its argument, and the null vector field $k_a = (dt)_a + (dr)_a$. Clearly, the energy momentum tensor satisfies the dominant energy condition due to $F' > 0$.

Specifically speaking, this model describes a family of concentric light shells with a flat Minkowski interior, ending with a final black hole of Schwarzschild radius $R = 2m$. The innermost light shell reaches the center at the time $t = 0$, and after the total duration of collapse Δ the outmost light shell finally arrives at the singularity. See Fig.2.

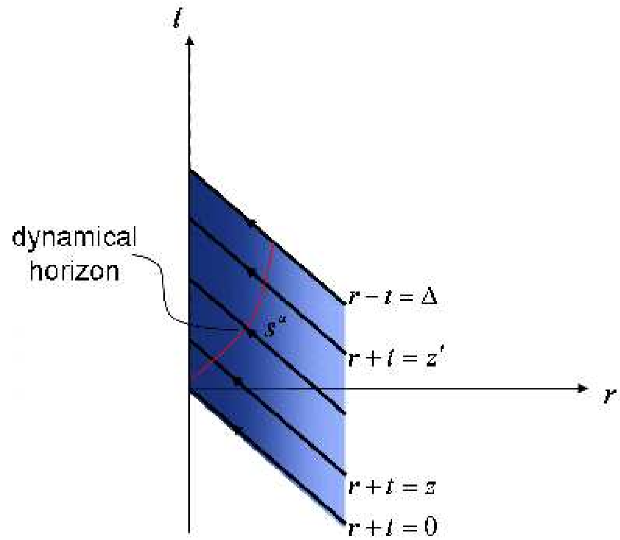

Figure 2: *a black hole is being formed by collapse of thick light shells between $r + t = 0$ and $r + t = \Delta$ in Eddington-Finkelstein coordinate, falling through the black hole dynamical horizon $r = 2mF$, which also serves as the infinite redshift surface.*

Next, to locate the black hole dynamical horizon in this model, let us first compute the initial expansion of the future-directed null normal to an arbitrary sphere characterized by some value of (t, r). The outgoing and ingoing null normals to these spheres can be chosen to be, respectively,

$$l^a = (1 + u)(\frac{\partial}{\partial t})^a + (1 - u)(\frac{\partial}{\partial r})^a, n^a = k^a = (\frac{\partial}{\partial t})^a - (\frac{\partial}{\partial r})^a, \tag{3.4}$$

then the corresponding expansions can be obtained as

$$\theta_l = \frac{2(1 - u)}{r}, \theta_n = -\frac{2}{r}. \tag{3.5}$$

Obviously, it follows from the definition presented in Sec.2 that the hyper-surface $u = 1$ is a black hole dynamical horizon if and only if its normal vector field is time-like, i.e.,

$$g^{ab}\nabla_a u \nabla_b u|_{u=1} = 2\frac{\partial u}{\partial t}(\frac{\partial u}{\partial r} - \frac{\partial u}{\partial t})|_{u=1} < 0. \tag{3.6}$$

Thus, according to Eq.(3.2), it is easy to find that the black hole dynamical horizon here is only the hyper-surface $u = 1$ within light shells, i.e., between the hyper-surfaces $r + t = 0$ and $r + t = \Delta$, as shown in Fig.2.

To proceed, we further assume that the collapse of light shells is adiabatical. Therefor the conserved entropy current of light shells can be written as

$$s^a = \frac{s'(\frac{r+t}{\Delta})}{4\pi\Delta r^2} k^a, \tag{3.7}$$

where the derivative of a function s with respect to its argument $s' > 0$.

We shall now check whether the generalized second law of thermodynamics is satisfied for the black hole dynamical horizon. As demonstrated in Fig.2, let $z' > z$, then the area difference of apparent horizons lying in $r + t = z$ and $r + t = z'$ reads

$$\delta A = 16\pi m^2 [F^2(\frac{z'}{\Delta}) - F^2(\frac{z}{\Delta})]. \tag{3.8}$$

On the other hand, by the conservation of the entropy current and Gauss theorem, the entropy flux S through the black hole dynamical horizon between the above apparent horizon is equal to that through the space confined within $z - t_p < r < z' - t_p$ at a time t_p in the distant past. Note that in the distant past, the light shells resided in asymptotically flat region. Thus by Eq.(3.3), the effective mass of light shells between $r + t = z$ and $r + t = z'$ can be obtained as

$$M_{eff} = m[F(\frac{z'}{\Delta}) - F(\frac{z}{\Delta})], \tag{3.9}$$

which equals the mass of the final black hole formed by collapse of these light shells. So employing Eq.(1.1), we have an upper bound on the entropy flux S

$$S \leq 4\pi m^2 [F(\frac{z'}{\Delta}) - F(\frac{z}{\Delta})]^2. \tag{3.10}$$

Combining Eq.(3.8) with Eq.(3.10), we have

$$S \leq \frac{\delta A}{4}, \tag{3.11}$$

which confirms the generalized second law of thermodynamics for the black hole dynamical horizon.

4. Discussions

we have proposed a new generalized second law of thermodynamics based on the notion of a black hole dynamical horizon. Its validity has also been demonstrated in a physically reasonable model of black hole formation by adiabatical collapse of thick light shells.

As mentioned in the beginning, along with the first and second laws of black hole mechanics for the dynamical horizon , our result further implies the black hole dynamical horizon may also have an interpretation of thermodynamics, especially one quarter of area of the black hole dynamical horizon may be identified with its entropy. It is therefore interesting to analyze if a derivation of the black hole entropy is available for the dynamical horizon based on the counting of micro-states in quantum gravity such as causal set theory, loop quantum gravity and string theory[9, 10, 11, 12, 13].

Even if it turns out that the black hole dynamical horizon has no interpretation of thermodynamics in an underlying quantum theory of gravity, our proposal can still be viewed as a covariant entropy bound conjecture on the dynamical horizon. It is noteworthy that its validity has also been verified in the cosmological context no matter whether the dynamical horizon is space-like or not[15]. Thus it is natural to expect that our proposal as a covariant entropy bound holds for the time-like analog of the black hole dynamical horizon.

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